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HEAT AND MOMENTUM TRANSFER ANALOGY FOR INCOMPRESSIBLE TURBULENT BOUNDARY LAYER FLOW

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THE ELEMENTARY surface renewal and penetration model is a relatively new turbulent transport model which has been enhanced by recent experimental evidence [8] indicating disturbances within the so-called laminar sublayer. This model is based on the hypothesis that fluid elements intermittently move from the turbulent core to the transport surface. Significantly, the surface renewal and penetration model has been found to be capable of correlating turbulent heat and momentum transfer data for hydrodynamically fully developed pipe flow [3, 12–14]. The purpose of this note is to illustrate the usefulness of this model in the analysis of turbulent transport processes associated with incompressible turbulent boundary layer flow.

Heat-transfer analyses for incompressible turbulent boundary layer flow generally have been based on the classical eddy diffusivity concept. As has been the case for internal flow problems, difficulties encountered in the evaluation of ε_H have posed the greatest problems. The simplifying assumptions for ε_H which have usually been employed (e.g. $\varepsilon_H = \varepsilon_m$) have restricted the usefulness of these analyses to fluids with values of the Prandtl number of the order of unity. Unfortunately, these type models provide little insight into the actual turbulent transport process.

The surface renewal and penetration model is based on the notion that molecular transfer is predominant during the time fluid elements are in the vicinity of the surface. As a result, the following system of equations is said to approximately define the molecular momentum transport mechanism within fluid elements in contact with the transport surface [2, 3]

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial \theta} \tag{1}$$

$$u = U_i \text{ at } \theta = 0$$
$$u = 0 \text{ at } y = 0$$
$$u = U_i \text{ at } y = \infty$$

where the instantaneous contact time, θ, begins at the instant a fluid element comes into contact with the surface and U_i is the axial velocity of fluid elements prior to contact with the surface. This system of equations appears to be applicable to turbulent boundary-layer flow as well as to turbulent flow in tubes. The initial velocity is assumed to be approximately equal to the free stream velocity, U_∞.

The solution of this system of equations leads to an expression for the local instantaneous velocity profile, u, within individual fluid elements at the wall of the form

$$\frac{u}{U_\infty} = \operatorname{erf}\left(\frac{y}{2\sqrt{(\nu\theta)}}\right). \tag{2}$$

Accordingly, an expression may be written for the local mean velocity profile within the wall region, $\bar{u}$, as

$$\bar{u} = \int_0^\infty u\, \phi(\theta)\, d\theta \tag{3}$$

where the contact time distribution, $\phi(\theta)$, is defined such that the product $\phi(\theta)\, d\theta$ represents the fraction of the surface with instantaneous contact time between θ and $\theta + d\theta$. Equations (2) and (3) coupled with a random contact time distribution function proposed by Danckwerts [1] of the form

$$\phi(\theta) = \frac{1}{\tau} \exp\left(\frac{-\theta}{\tau}\right) \tag{4}$$

lead to a relationship for the local mean wall shear stress, σ_{0x}, of the form

$$\sigma_{0x} = \mu \frac{\partial \bar{u}}{\partial y}\bigg|_0 = \rho U_\infty \sqrt{\left(\frac{\nu}{\tau}\right)}. \tag{5}$$

The prediction for the mean transport flux has been shown to be fairly insensitive to the form of the distribution function (3); the mean residence time, τ, represents the mean length of time fluid elements remain in the wall region. Equation (5) may be rearranged and written in terms of the local mean friction factor, f_x, as

$$f_x = \frac{2}{U_\infty}\sqrt{\left(\frac{\nu}{\tau}\right)}. \quad (6)$$

This expression may be solved for τ in terms of f_x. Knudson and Katz [17] have suggested an expression for f_x of the form

$$f_x = 0{\cdot}0585\, Re_x^{-0{\cdot}2} \quad (7)$$

which is in agreement with experimental data for values of the Reynolds number less than 2×10^6.

Perhaps the major distinction between the present application and the problem of hydrodynamically fully developed flow in a tube lies in the analysis of the mean frequency of renewal, $1/\tau$. Because the tendency for fluid elements to move from the turbulent core to the wall region is a function of the hydrodynamic parameters of the system, $1/\tau$ appears to be independent of the axial coordinate, x, for fully developed flow. Such is not the case, however, for turbulent boundary layer flow.

The application of the basic surface renewal and penetration model to heat transfer associated with turbulent boundary layer flow is synonymous to the above analysis. A similar analysis has recently been presented in more detail for turbulent tube flow of liquid metals and moderate Prandtl number fluids [12]. This analysis involves the solution of the one-dimensional unsteady energy equation with boundary conditions of the form $t(0, y) = T_i$, $t(\theta, \infty) = T_i$ and $t(\theta, 0) = T_0$ for uniform wall temperature, T_0, or $\partial t(\theta, 0)/\partial y = -q_0/k$ for uniform wall flux, q_0; T_i is the temperature of fluid elements prior to contact with the surface and k is the thermal conductivity. The solution for the local instantaneous temperature profile, t, coupled with the age distribution principle gives rise to an expression for the local mean coefficient of heat transfer, h_x, of the form

$$h_x = \frac{T_0 - T_i}{T_0 - T_\infty}\frac{k}{\sqrt{(\alpha\tau)}} \quad (8)$$

for both uniform wall temperature and uniform wall heat flux conditions; α is the thermal diffusivity. T_i may be fairly well approximated by the free stream temperature, T_∞, for fluids other than liquid metals [12].

Due to the assumption that eddies move into direct contact with the surface, the present analysis must be restricted to fluids with low to moderate values of the Prandtl number, Pr. Based on the experimental work by Popovich and Hummel [8], turbulent eddies do not come into direct contact with the surface. In this regard, the effect on the mean transfer rate of eddies moving to within various distances of the surface has been accounted for by Harriott [4] and Thomas [14]; these surface rejuvenation models correlate experimental data for turbulent tube flow for $0{\cdot}5 < Pr < 10^5$. Importantly, these models reduce to the simple surface renewal and penetration model for moderate to low values of the Prandtl number, suggesting the proportionality $Nu \sim \sqrt{(Pr)}$. In contrast, the surface rejuvenation models lead to the familiar proportionality $Nu \sim Pr^{1/3}$ for fluids with values of the Prandtl number very much greater than unity.

With τ defined by equation (6) in terms of f_x, equation (8) gives rise to the following simple analogical expression for the local mean Nusselt number (with T_i set equal to T_∞).

$$Nu_x = \frac{f_x}{2} Re_x \sqrt{Pr}. \quad (9)$$

This analogy between the heat (or mass) and momentum transfer is a result of the renewal mechanism. That is, fluid elements are visualized as moving to the surface with molecular heat and momentum transfer occurring during the residence time. Hence, the mean residence time becomes identical for both and thus provides the connecting link between turbulent heat and momentum transfer.

With f_x defined by equation (7), equation (9) takes the form

$$Nu_x = 0{\cdot}0292\, Re^{0{\cdot}8}\, Pr^{0{\cdot}5}. \quad (10)$$

This expression is comparable to the following semi-empirical expression offered by Kays [6] for the Prandtl number range of gases.

$$Nu_x = 0{\cdot}0295\, Re^{0{\cdot}8}\, Pr^{0{\cdot}6}. \quad (11)$$

Equation (10) is compared with experimental data for air (uniform wall temperature) by Seban and Doughty [9] and Reynolds *et al.* [11] in Figs. 1 and 2. The correlation of these

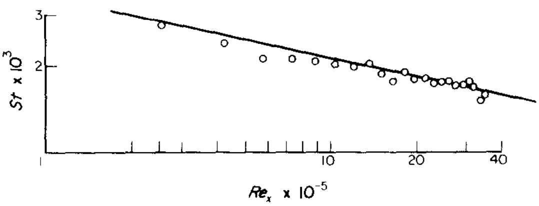

FIG. 1. Comparison of equation (6) with experimental data by Seban and Doughty [9] for flat plate boundary-layer flow of air and uniform wall temperature.

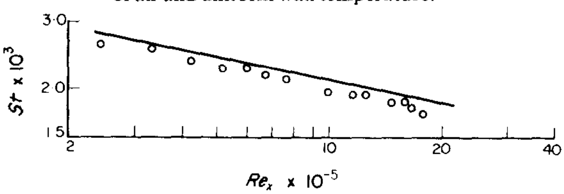

FIG. 2. Comparison of equation (6) with experimental by Reynolds *et al.* [11] for flat plate boundary-layer flow of air and uniform wall temperatures.

data appears to be excellent. The present analysis suggests that the heat transfer flux is essentially independent of the form of the axial boundary condition, except for heat transfer to low turbulence intensity liquid metal flow [12]. Accordingly, equation (10) is compared with data for air by Reynolds *et al.* for a ramp wall temperature distribution in Fig. 3. Again the agreement between theory and data is good.

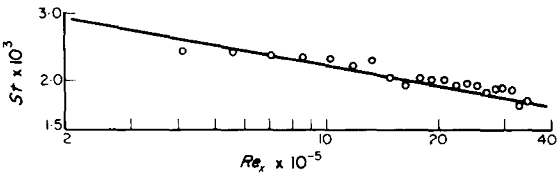

FIG. 3. Comparison of equation (6) with experimental data by Reynolds *et al.* [11] for flat plate boundary-layer flow of air and a ramp wall temperature.

Based on the present analysis, the overall mean Nusselt number for turbulent flow past a flat plate of length L becomes

$$Nu_L = Re_L \sqrt{Pr} \int_0^1 \frac{f_x}{2} \, d\left(\frac{x}{L}\right). \tag{8}$$

For simplicity this analysis neglects the existence of the laminar boundary layer and therefore is valid only when L is much greater than this laminar region. This expression is compared with experimental data by Slegel [10] for the turbulent flow of air past a flat plate in Fig. 4. The simple analogy between heat and momentum transfer is seen to be in excellent agreement with the experimental data. The analogy is also shown in Fig. 4 to be in good agreement with heat-transfer data for the flow of air parallel to a circular cylinder by Jakob and Dow [5].

To conclude, the surface renewal and penetration model appears to provide a clear theoretical basis for expressing an analogy between heat and momentum transfer for turbulent boundary-layer flow, as well as for turbulent flow in tubes. Unlike the somewhat artificial turbulence models which have been based upon the eddy diffusivity concept, this model provides a reasonable picture of the actual turbulent transport mechanism. Although the present analysis must be restricted to fluids with moderate values of the Prandtl number ($0{\cdot}5 < Pr < 5{\cdot}0$), the sound basis upon which this model is based provides the potential for developing relationships for the very low and high Prandtl number regions.

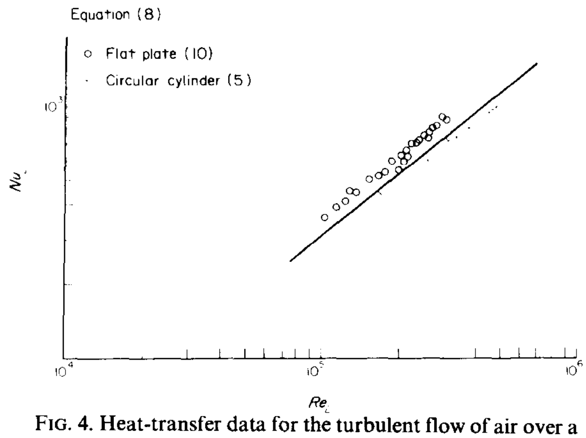

FIG. 4. Heat-transfer data for the turbulent flow of air over a flat plate and a circular cylinder.

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